Quasi-steady dissipative nonlinear critical layer in a stratified shear flow

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When a wave with small but finite amplitude ε propagates towards the CL, where the effects of nonlinearity and dissipation are essential, the jump of mean vorticity over the CL appears. For the dynamically stable stratified shear flow with the gradient Richardson number $Ri>1/4$ the jump of vorticity has the same order as the undisturbed one [J. Fluid Mech. **233**, 25 (1991)]. The process of formation of the flow with this substantial jump of vorticity (or ''break'' of the velocity profile) in the CL is studied at large time after beginning of the process. The transition region between the CL and the undisturbed flow, the dissipation boundary layer (DBL), is shown to be formed. Its thickness grows in time proportional to $\sqrt{t}$ (t being time), and the CL moves towards the incident wave. When the jump of the wave momentum flux over the CL is constant in time, the flow characteristics can be found in the most simple way. The velocity profile in the DBL appears to be self-similar, the displacement of the CL is proportional to $\sqrt{t}$ and the values of vorticity at the both sides of the CL do not depend on time and they are determined only by the constant wave momentum flux. It is shown that, to provide the constant jump of the wave momentum flux the amplitude of the wave radiated by the source in the undisturbed flow region should vary in a certain complicated manner, because it reflects from the time-dependent (broadening) velocity profile in the DBL. On the other hand, the wave momentum flux from the steady source (for example, the corrugated wall) depends on time. When the coefficients of reflection from the CL (R) and from the DBL (r) are small, this dependence is weak and the wave and flow parameters depending on time are found as series in R and r. The wave–flow interaction for this case is studied.
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I. INTRODUCTION

The investigation of singular wavelike disturbances of small amplitude ε superimposed on shear flows leads to the problem of removing singularities in equations for the disturbances. The singularities occur at the critical points, where the phase velocity of the perturbation coincides with the flow velocity. One of the approaches to solving the problem is taking into account dissipation and nonlinearity in the critical layer (CL)—the small vicinity around the critical point. It corresponds to the result of evolution of the flow at large time. This approach was employed in the works by Haberman[1,2] for homogeneous and slightly stratified shear flows and in the paper by Troitskaya[3] for the flow with the gradient Richardson number $Ri>1/4$. The last work further is called **I** for short. It is shown in the works cited, that the combined effect of nonlinearity and dissipation leads to deformation of the mean velocity profile, namely a jump of the mean vorticity across the CL arises, i.e. for $|z|\to\infty$ the vorticity averaged over the wave period,

$$\frac{\langle\omega\rangle}{\Omega_0}=\left(1+\Gamma+\frac{\Delta\Gamma}{2}\,\mathrm{sign}\,(z-z_c)\right), \tag{1}$$

denoting Ω_0 the undisturbed vorticity, z the vertical coordinate, z_c the CL coordinate $\Delta\Gamma$ is the normalized vorticity jump, Γ is a constant discussed below. The value of $\Delta\Gamma$ appears to be larger in order than the disturbance of order ε causing it. For the logarithmic singularity in the wavelike disturbance of a homogeneous flow[1] and the Rossby wave on the β-plane[6] $\Delta\Gamma$ is of order $\varepsilon^{1/2}$. For the algebraic branch point in the stratified shear flow the value of $\Delta\Gamma$ is of unity order for $Ri>1/4$ (**I**).

The mean velocity disturbance related to the vorticity (1) grows to infinity with the distance from the CL. To limit this growth one must come out of the framework of the steady approximation and take into account time evolution as it was supposed in the works by Maslowe.[4,5] For example, the unsteady process of vorticity diffusion from the CL due to viscosity leads to arising of diffusion layers, where the mean flow distortion drops with the distance from the CL. This approach was employed for the Rossby wave CL,[6] when the mean flow distortion is the small value of order $\varepsilon^{1/2}$. The similar diffusion layers was obtained[7,8] in the weak-nonlinear approximation for the stratified shear flow with the Richardson number $Ri\approx 1/4$, when the deformation of the mean velocity profile and the jump of vorticity are not large. In the present work, the stratified shear flow with $Ri>1/4$ is considered, when the vorticity jump has the same order as the vorticity of the undisturbed flow. It leads to some specific properties of the flow in the diffusion layers in comparison with the homogeneous flow and the stratified flow with $Ri\approx 1/4$. In particular, in the process of the unsteady deformation of the velocity profile, the CL is moving towards the incident wave proportionally to $\sqrt{t}$ (where t is the time from the start of the process). The similar displacement of the CL was obtained in the numerical experiment.[9]

It should be taken into account that the vorticity jump $\Delta\Gamma$ in (1) can be obtained within the steady approximation, but the quantity Γ is the arbitrary constant of the stationary problem, i.e. in the steady approximation one can find only the difference $(\Gamma_+ - \Gamma_-)$, but not the values of vorticity on both sides of the CL $(\Gamma_+ = 1+\Gamma+\Delta\Gamma/2$ and $\Gamma_- = 1+\Gamma-\Delta\Gamma/2)$. Γ_+ and Γ_- can be obtained only within the unsteady problem and the result depends on the boundary conditions in the outer region of the CL. In the present work, Γ_+ and Γ_- are found for the unbounded shear flow, when disturbances decrease at infinity. It should be mentioned that for the Rossby waves on the zonal shear flow when $\Delta\Gamma$ is small value of order $\varepsilon^{1/2}$, the solution to the initial problem gives $\Gamma=0$[6] and $\Gamma_\pm = 1\pm\Delta\Gamma/2$. In the present case Γ is shown to be not equal to zero.

In section II, the main equations for the problem are formulated. The qualitative properties of the flow in the CL vicinity and the basic approximations are discussed in section III. In section IV the process of diffusion of the mean vorticity from the steady source posed in the CL is studied. The wave–flow interaction is investigated in section V. In section VI the numerical model employed for the investigation of the CL is briefly described and the dependence of the CL parameters on the inverse inner Reynolds number in the CL is presented. They are similar to the results obtained in **I**. Finally, the example of deformation of the mean velocity profile due to interaction with the waves generated by the stratified shear flow over a corrugated surface is considered in section VII.

II. THE BASIC EQUATIONS

Consider the stratified shear flow with the undisturbed velocity and density vertical profiles $V_0(z_d)$ and $\rho_0(z_d)$, respectively (z_d being the dimensional vertical coordinate). Suppose that the gradient Richardson number, equal to $N^2/(dV_0/dz_d)^2 > 1/4$ everywhere in the flow [where $N^2 = -(g/\rho_0)(d\rho_0/dz_d)$ is the buoyancy frequency]. Consider the problem of wave propagation towards this flow. Suppose that the flow velocity coincides with the phase speed of the wave at some level. In the vicinity of this level the critical layer (CL) is formed, where strong wave–flow interaction takes place. In the process of the wave–flow interaction two stages can be specified. Flow evolution at the beginning stage depends on the amplitude and the shape of the front of the incident wave. If the amplitude is large and the front is sharp enough then the complicated nonlinear transition process takes place (high harmonic generation, wave induced instability, etc.). This process can be studied only in the framework of the numerical models.[10,11] After relaxation of the transition process due to viscosity the quasi-steady flow is established and the time evolution is determined by the slow diffusion process. In this case the flow in the CL vicinity is almost steady, and its dynamics is determined by balance of nonlinearity and dissipation. Within the Boussinesq approximation, the nondimensional equations for the vorticity and density are as follows:

$$\frac{\partial\omega}{\partial t}+\frac{\partial\omega}{\partial x}\frac{\partial\psi}{\partial z}-\frac{\partial\omega}{\partial z}\frac{\partial\psi}{\partial x}-\frac{g'}{\rho_0}\frac{\partial\rho}{\partial x}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial z^2}+\frac{\partial^2\omega}{\partial x^2}\right),$$
$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}\frac{\partial\psi}{\partial z}-\frac{\partial\rho}{\partial z}\frac{\partial\psi}{\partial x}=\frac{1}{Re}\frac{1}{Pr}\left(\frac{\partial^2\rho}{\partial z^2}+\frac{\partial^2\rho}{\partial x^2}\right), \tag{2}$$

and

$$\omega=\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial x^2}.$$

Here, x and z are nondimensional horizontal and vertical coordinates normalized on the scale of the mean flow L_0; t is nondimensional time normalized by the characteristic scale of the mean flow L_0/U_0; U_0 is the characteristic velocity of the mean flow; ψ is the nondimensional stream function scaled by L_0U_0; ω is the dimensionless vorticity scaled by U_0/L_0; ρ is the nondimensional density, normalized by its character value ρ_{00}; $Re=U_0L_0/\nu_0$ is the Reynolds number defined in terms of the parameters of the mean flow; $Pr=\nu_0/\nu_t$ is the Prandtl number; ν_0, ν_t are the viscosity and the thermoconductivity coefficients; $g'=gL_0/U_0^2$ is the dimensionless gravity.

Suppose that the harmonic wave of small amplitude radiated by an external source propagates towards the basic flow described above and the phase speed of the wave coincides with the flow velocity at some level. Then CL is formed in the vicinity of this level, where strong wave–flow interaction takes place. It is accompanied with slow variation in time of the mean flow and the wave amplitude and generation of the high harmonics of the basic disturbance. Then the solution to the system (2) can be searched as follows:

$$\psi=\int^z U(z,t)dz+\varepsilon\sum_{j=1}^{\infty}\mathrm{Re}\,\psi_j(z,t)e^{-ikj\varsigma},$$

where $\varsigma = x-ct$, c is the dimensionless phase velocity of the wave, k is the dimensionless wave number scaled by L_0. It should be emphasized that both the wave field amplitude and the mean velocity profile can depend on time.

The nonlinear system (2) can be linearized relative to the mean flow if the disturbance amplitude ε is small. Neglecting the dissipation and time dependence of the wave amplitude gives the Taylor–Goldstein equation for the complex amplitude of the basic disturbance of the stream function ψ_1,

$$\frac{d^2\psi_1}{dz^2}-\frac{d^2U/dz^2}{U-c}\psi_1+\left(\frac{(NL_0/U_0)^2}{(U-c)^2}-k^2\right)\psi_1=0. \tag{3}$$

The density disturbance ρ_1 is expressed by ψ_1 in the following way:

$$\rho_1=-\frac{(NL_0/U_0)^2}{U-c}\psi_1. \tag{4}$$

Equations (3) and (4) have the singularity points z_c, where $U(z_c)=c$ (z_c is the critical level). It means that the approximation neglecting dissipation, nonlinearity and time dependence is invalid in the CL-vicinity and these factors should be taken into account here. The regions where they are essential are called nonlinear, viscous and nonstationary CL, respectively. The scales of the CL are as follows:[12–15]

$$\delta_{vis}=(ReU_zk)^{-1/3}:\quad \delta_{nl}=\varepsilon^{2/3};$$
$$\delta_t=\left(\frac{|\partial\psi/\partial t|}{|\psi|}\right)(kU_z)^{-1}. \tag{5}$$

The nonlinear effects in the CL are investigated below, so the transformation from the dimensionless variables z,x,t,ψ,ω,ρ to the intrinsic variables of the nonlinear CL is made:

$\eta = z/\varepsilon^{2/3}$ is the ''inner'' vertical coordinate,
$\tau = kt\varepsilon^{2/3}\Omega_0$ is the ''inner'' time,
Ω_0 is the dimensionless vorticity value far from the CL,
$\xi = k\varsigma$ is the normalized intrinsic horizontal coordinate,
$\varphi = (\psi - cz)/\varepsilon^{4/3}\Omega_0$ is the inner stream function,
$\chi = \omega/\Omega_0$ is the normalized vorticity,
$b = (\rho - 1)/(\varepsilon^{2/3}(-d\rho_0/dz))$ is the ''inner'' density,

here $d\rho_0/dz$ is the characteristic dimensionless density gradient of the basic flow far from the CL.

In the inner variables the system (2) is as follows:

$$\frac{\partial \chi}{\partial \tau} + \frac{\partial \chi}{\partial \xi}\frac{\partial \varphi}{\partial \eta} - \frac{\partial \chi}{\partial \eta}\frac{\partial \varphi}{\partial \xi} - Ri\frac{\partial b}{\partial \xi} = \lambda\left(\frac{\partial^2 \chi}{\partial \eta^2} + \frac{\partial^2 \chi}{\partial \xi^2}\alpha^2\right),$$

$$\frac{\partial b}{\partial \tau} + \frac{\partial b}{\partial \xi}\frac{\partial \varphi}{\partial \eta} - \frac{\partial b}{\partial \eta}\frac{\partial \varphi}{\partial \xi} = \frac{\lambda}{Pr}\left(\frac{\partial^2 b}{\partial \eta^2} + \frac{\partial^2 b}{\partial \xi^2}\alpha^2\right), \qquad (6)$$

$$\chi = \frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \xi^2}\alpha^2,$$

where $\lambda = (kRe\varepsilon^2\Omega_0)^{-1}$ is the nonlinearity parameter, Ri is characteristic value of the Richardson number of the basic flow far from the CL. It obviously follows from (6) and λ is the inverse value of the Reynolds number in the CL defined in terms of the vertical coordinate. Taking (5) into account gives $\lambda = (\delta_{vis}/\delta_{nl})^3$. Here $\alpha = k\varepsilon^{2/3}$ is the ratio of the nonlinear CL scale to the horizontal wavelength. Further, the wave amplitude ε is supposed to be small enough, so $\alpha \ll 1$, and the terms $\alpha^2\partial^2 f/\partial\xi^2$ (where $f = \{\chi, b, \psi\}$) can be neglected considering flows in the vicinity of the CL with the characteristic scale of order δ_{nl} (5). But far from the CL the vertical scale of the disturbances increases and these terms can become essential.

The investigation of the evolution of the flow in the CL vicinity will be made within the system (6). Since the disturbance is supposed to be periodically varying in the coordinate ξ, the hydrodynamic fields can be represented as the sum of two components, namely, one that is averaged over the disturbance period and one that is ξ-dependent (periodic in space), i.e.

$$\varphi = \varphi_0(\eta,\tau) + \varphi'(\eta,\xi,\tau),$$

$$b = b_0(\eta,\tau) + b'(\eta,\xi,\tau),$$

$$\chi = \chi_0(\eta,\tau) + \chi'(\eta,\xi,\tau).$$

Far from the CL, φ_0, b_0, χ_0 tend to undisturbed values and φ', b', χ' have the sense of the wave disturbance. First, consider the evolution of the average fields.

III. THE QUALITATIVE PROPERTIES OF THE FLOW IN THE CL VICINITY. THE BASIC APPROXIMATIONS

The equations for the average vorticity χ_0 and density b_0 fields can be obtained by averaging of the first and second equations of the system (6) over the period of disturbances. The transformation gives

$$\frac{\partial \chi_0}{\partial \tau} - \lambda\frac{\partial^2 \chi_0}{\partial \eta^2} = -\frac{1}{2\pi}\frac{\partial^2}{\partial \eta^2}\int_0^{2\pi}\varphi_\eta\varphi_\xi d\xi, \qquad (7)$$

$$\frac{\partial b_0}{\partial \tau} - \frac{\lambda}{Pr}\frac{\partial^2 b_0}{\partial \eta^2} = -\frac{1}{2\pi}\frac{\partial}{\partial \eta}\int_0^{2\pi} b\varphi_\xi d\xi,$$

$$\chi_0 = \frac{\partial^2 \varphi_0}{\partial \eta^2}. \qquad (8)$$

Integrating (7) with respect to η gives the equation for the mean horizontal velocity,

$$\frac{\partial u_0}{\partial \tau} - \lambda\frac{\partial^2 u_0}{\partial \eta^2} = -\frac{1}{2\pi}\frac{\partial}{\partial \eta}\int_0^{2\pi}\varphi_\eta\varphi_\xi d\xi. \qquad (9)$$

The source in the diffusion equation (9) is the radiation force,

$$F_{rad} = \frac{\partial T}{\partial \eta},$$

where

$$T = -\frac{1}{2\pi}\int_0^{2\pi}\varphi_\eta\varphi_\xi d\xi \qquad (10)$$

is the vertical flux of the horizontal momentum (the radiation stress or the Reynolds stress). A simple physical interpretation of the equation (9) can be proposed. It is the equation of motion of a liquid particle averaged over the disturbance period, where the averaged acceleration is determined by the radiation and viscous forces. The form of the equation (8) for the average density is similar to (9). Equation (8) is the diffusion equation in which the source is equal to the first derivative of the vertical mass flux,

$$B = -\frac{1}{2\pi}\int_0^{2\pi} b\varphi_\xi d\xi. \qquad (11)$$

It obviously follows from (8) that variation of the average density at some level is determined by the ''flowing into and out'' mass fluxes and by diffusion of the density.

To complete the formulation of the problem, the initial and boundary conditions for the partial differential equations (7), (8), (9) should be given. At the initial moment the velocity and density profiles are supposed to be linear, i.e.

$$u_0 = \eta; \quad b_0 = -\eta.$$

At infinity ($\eta \to \pm\infty$) the distortions of the average profiles are supposed to vanish—

$$u_0(\eta \to \pm\infty, \tau) = \eta,$$

$$b_0(\eta \to \pm\infty, \tau) = -\eta. \qquad (12)$$

First, consider qualitatively the evolution of the average flow at large time using the qualitative properties of the equation (9). Suppose that the time variation of the flow in the CL vicinity is determined only by the diffusion process.

This supposition is correct, because the flow is stable. (No processes more rapid than diffusion are discussed below.) In this case the CL becomes quasi-stationary at a sufficiently large time period, more exactly large in comparison with the diffusion time at the scale of the CL:

$$\tau \gg \frac{\delta^2}{\lambda}. \tag{13}$$

The scale of the quasi-stationary CL is determined by the viscosity or nonlinearity, it is of order $\lambda^{1/3}$ in the first case and of order 1 in the second case, i.e.

$$\delta = \max\{\lambda^{1/3}, 1\}. \tag{14}$$

Expression (13) is the necessary (but not sufficient) condition of quasi-stationarity of the CL. The conditions of quasi-stationarity are discussed below in detail.

Since diffusion is a decelerating process, the CL becomes more ''stationary'' with time. The wave momentum flux is known to be constant in the quasi-stationary waves. This means that $\partial T/\partial\eta \neq 0$ only in the CL-vicinity of width δ, i.e. only here the radiation force differs from zero. In the quasi-stationary CL the radiation force should be balanced by the viscous force, so the jump of vorticity across the CL appears as in the stationary CL (see **I**). Suppose that the wave propagates towards the CL in the negative direction of the vertical axes like in **I**. Then the direction of the radiation force in the CL vicinity determines that in the region $\eta > 0$ the value of the mean vorticity is greater than the undisturbed one and in the region $\eta < 0$ it is less than the undisturbed one. And there is no disturbance of vorticity far from the CL (for $\eta \to \pm\infty$) [see (12)]. Diffusion of vorticity from the CL occurs due to viscosity. So the transition region from the CL to the undisturbed flow is forming. Further, it is called the diffusion boundary layer (DBL). It obviously follows from the equation (9) that the scale of the diffusion spreading of the flow in DBL is the diffusion length $\sqrt{\lambda\tau}$, i.e. it grows in time.

The critical level is the point at the velocity profile, where the flow velocity is constant and equal to the phase velocity of the wave. At this point, there is the ''break'' of the velocity profile. [The scale of this break δ is nonzero, but small in comparison with the diffusion length $\sqrt{\lambda\tau}$ according to (13).] This profile can be realized only if the CL is moving in the positive direction of η, i.e. towards the incident wave momentum flux. The similar displacement was obtained in the numerical experiment.[9] Although the law of motion of CL was more complicated, namely, acceleration occurred at the beginning and deceleration later, that behavior took place because the beginning stage of the wave–flow interaction in the CL was studied, when the time from the start of the process was less than the time of diffusion at the scale of the CL (see Ref. 9). It should be mentioned that according to the review article by Stewartson[16] there is only one mesh point on the CL in these calculations. But the width of the nonlinear and viscous CL was about 20 m and the grid step $3\frac{1}{3}$ m. Our control calculations show that the computation error is rather small for this relation between the grid step and the CL scale.

Now we will obtain quantitatively these qualitative properties. In general the law of the CL motion can be presented as follows:

$$\eta = s(\tau)\sqrt{\lambda\tau}, \tag{15}$$

here $s(\tau)$ is an unknown function to be determined.

In the vicinity of the moving CL the natural coordinate is the ''intrinsic'' one,

$$h = \eta - s(\tau)\sqrt{\lambda\tau}.$$

Considering diffusion processes, it is convenient to use the quantity

$$\nu = \frac{1}{2\sqrt{\lambda\tau}}, \tag{16}$$

i.e. the inverse diffusion length instead of time.

The hydrodynamic equations (6) in the variables h, ξ, ν (instead of η, ξ, τ) are expressed in the following way:

$$\overbrace{-2\lambda\frac{\partial\chi}{\partial\nu}\nu^3}^{I}-\frac{\partial\chi}{\partial h}\overbrace{\left(\lambda s\nu-\lambda\nu^2\frac{ds}{d\nu}\right)}^{II}+\overbrace{\frac{\partial\chi}{\partial\xi}\frac{\partial\varphi}{\partial h}-\frac{\partial\chi}{\partial h}\frac{\partial\varphi}{\partial\xi}-Ri\frac{\partial b}{\partial\xi}}^{III}=\lambda\left(\frac{\partial^2\chi}{\partial h^2}+\frac{\partial^2\chi}{\partial\xi^2}\alpha^2\right),$$

$$\overbrace{-2\lambda\frac{\partial b}{\partial\nu}\nu^3}^{I}-\overbrace{\frac{\partial b}{\partial h}\left(\lambda s\nu-\lambda\nu^2\frac{ds}{d\nu}\right)}^{II}+\overbrace{\frac{\partial b}{\partial\xi}\frac{\partial\varphi}{\partial h}-\frac{\partial b}{\partial h}\frac{\partial\varphi}{\partial\xi}}^{III}=\frac{\lambda}{Pr}\left(\frac{\partial^2 b}{\partial h^2}+\frac{\partial^2 b}{\partial\xi^2}\alpha^2\right),$$

$$\chi=\frac{\partial^2\sigma}{\partial h^2}+\frac{\partial^2\sigma}{\partial\xi^2}\alpha^2. \tag{17}$$

At large time periods when (13) is valid there are two strongly different scales δ, defined by (14), and $1/\nu$ [ν is defined by (16)] in the system (17), so it can be solved by the method of matched asymptotic expansions in the small parameter $\delta\nu$ equal to the relation of the CL width to the diffusion scale at large time period. Because δ is of order unity within a wide range of the parameter λ, the expansion is made in the parameter ν.

A. Inner solution

The zeroth order in ν corresponds to the quasi-stationary approximation when all the terms containing ν-products can be omitted. The system (17) enables us to obtain some additional limitations on using of the quasistationary approximation in the CL vicinity. This approximation is valid if the terms I and II are small in comparison with III and the right hand sides (RH). Namely

$$\text{I} \ll \text{III}, \quad \lambda\nu^2 \ll h, \quad \tau \gg 1/h, \tag{18a}$$

$$\text{I} \ll \text{RH}, \quad \lambda\nu^2 \ll \lambda/h^2, \quad \tau \gg h^2/\lambda, \tag{18b}$$

$$\text{II} \ll \text{III}, \quad \lambda s\nu/h \ll h, \quad \sqrt{\lambda\tau} \gg \lambda s/h^2, \tag{18c}$$

$$\text{II} \ll \text{RH}, \quad \lambda s\nu/h \ll \lambda/h^2, \quad \tau \gg h^2/\lambda s^2. \tag{18d}$$

The inequalities (18) are written in both variables ν and τ.

Let us discuss the sense of inequalities (18). First, it should be taken into account that the terms I describe time evolution in the frame of reference moving according to (15), and the terms II describe the influence of variable bulk motion. The characteristic time scale of the terms I is τ, and the time scale of the terms II is the time of the CL displacement from the point η to the point $\eta+h$, which is approximately equal to $h\sqrt{\tau}/s\sqrt{\lambda}$ [see (15)]. The terms III describe inertial motion with the characteristic time scale $1/h$ and the right hand (RH) describes the diffusion at the distance h, which has the characteristic time h^2/λ. The inequalities (18a) and (18b) mean that the characteristic time after beginning of the process τ is large in comparison with the characteristic inertia time and the time of diffusion at the distance of order h. The inequalities (18c) and (18d) mean that the characteristic time of the CL displacement at the distance h is large in comparison with the characteristic times of inertia and diffusion at the distance h. Since translation of the CL is the decelerated motion, it does not influence the flow in the CL vicinity at sufficiently large time. The opposite case was studied by Haynes and Cowley[18] when the uniform motion of the inviscid CL for the Rossby wave dramatically changes its inner flow.

If the inequalities (18) are valid the solution of the system (17) can be expressed as a series in ν;

$$(\chi,b,\varphi)=(\chi^{(0)},b^{(0)},\varphi^{(0)})+\nu(\chi^{(1)},b^{(1)},\varphi^{(1)})\cdots .$$

In the zeroth order of approximation in ν, the system (17) is as follows:

$$\frac{\partial\chi^{(0)}}{\partial\xi}\frac{\partial\varphi^{(0)}}{\partial h}-\frac{\partial\chi^{(0)}}{\partial h}\frac{\partial\varphi^{(0)}}{\partial\xi}-Ri\,\frac{\partial b^{(0)}}{\partial\xi}=\lambda\,\frac{\partial^2\chi^{(0)}}{\partial h^2},$$
$$\frac{\partial b^{(0)}}{\partial\xi}\frac{\partial\varphi^{(0)}}{\partial h}-\frac{\partial b^{(0)}}{\partial h}\frac{\partial\varphi^{(0)}}{\partial\xi}=\frac{\lambda}{Pr}\frac{\partial^2 b^{(0)}}{\partial h^2}, \tag{19}$$
$$\chi^{(0)}=\frac{\partial^2\varphi^{(0)}}{\partial h^2}.$$

The system (19) was considered already in **I**.

For $|h|\gg\delta$ (but $|h|\ll\sqrt{\lambda\tau}$), the profiles of vorticity and the Brunt–Vâisâlâ frequency averaged over a period of disturbances are constant. Then the wave disturbances can be easily found from the Taylor–Goldstein equation by the Frobenius method. As a result, the asymptotic solution of (19) for $\delta\ll|h|\ll\sqrt{\lambda\tau}$ is as follows:

$$\varphi_{\pm}^{(0)}=hc_{\pm}+\frac{\Gamma_{\pm}h^2}{2}+\sum_{n=1}^{\infty}\mathrm{Re}([A_{\pm}^{(n)}|h|^{1/2+i\mu_{\pm}}+B_{\pm}^{(n)}|h|^{1/2-i\mu_{\pm}}]e^{in\xi}), \tag{20a}$$

$$b_{\pm}^{(0)}=\beta_{\pm}-N_{\pm}^2h\mp\frac{N_{\pm}^2}{\Gamma_{\pm}}\sum_{n=1}^{\infty}\mathrm{Re}([A_{\pm}^{(n)}|h|^{-1/2+i\mu_{\pm}}+B_{\pm}^{(n)}|h|^{-1/2-i\mu_{\pm}}]e^{in\xi}), \tag{20b}$$

where $\mu_{\pm}=\sqrt{Ri/\Gamma_{\pm}^2-1/4}$.

The following expression for the values of the jump of the vorticity $(\Gamma_{+}-\Gamma_{-})$ across the CL can be found as in **I**:

$$2\lambda(\Gamma_{+}-\Gamma_{-})=T(+\infty)-T(-\infty). \tag{21a}$$

Using (20) it can be easily obtained from (10), that

$$T(\pm\infty)=\mu_{\pm}\sum_{n=1}^{\infty}(|A_{\pm}^{(n)}|^2-|B_{\pm}^{(n)}|^2).$$

The jump of the density gradient across CL is absent, and

$$N_{+}^2=N_{-}^2=1. \tag{21b}$$

Here the problem of harmonic wave propagation towards the CL is studied, so $A_{+}^{(1)}\neq 0$. The amplitudes of other harmonics can be found from the solution of the set (19) (see Sec. VI). It should be mentioned, however, as was demonstrated in **I**, for moderate Richardson numbers the amplitudes of high harmonics were very small in comparison with the amplitude of the fundamental harmonic for $|h|\gg\delta$. So further they will be neglected in this region. Note that the similar property of high harmonics to be small in comparison with the fundamental one was demonstrated in other different kinds of flows with CL. So Smith and Bodonyi[17] show the insignificant role of high harmonics for homogeneous flow with the logarithmic singularity in the CL. For the marginally stable stratified shear flow with the algebraic branch point in the CL calculation carried out by Churilov and Shukhman[8] revealed that the contribution of the second harmonic to the Landau constant in the Landau–Stewart equation was 5 times smaller than the contribution of the mean flow, i.e. the nonlinear effect was determined mainly by the interaction of the fundamental harmonic and the mean flow.

B. Outer solution

Consider now the region of the diffusion boundary layer, where $h \sim \sqrt{\lambda\tau}$ i.e., the conditions (18b) and (18d) are not valid. The outer vertical variable $H=\nu h$ is natural for this region. The inner solution (20a) and (20b) expressed in the variable H has the following form:

$$\varphi_{\pm}^{(0)} = \frac{1}{\nu^2}\left[\frac{\Gamma_{\pm}H^2}{2}\ \nu H c_{\pm} + \nu^{3/2}\sum_{n=1}^{\infty} \mathrm{Re}([A_{\pm}^{(n)}|H|^{(1/2)+i\mu_{\pm}}\nu^{-i\mu_{\pm}} + B_{\pm}^{(n)}|H|^{(1/2)-i\mu_{\pm}}\nu^{i\mu_{\pm}}]e^{in\xi})\right], \tag{22a}$$

$$b_{\pm}^{(0)} = \frac{1}{\nu}\left[-N_{\pm}^2 H + \nu\beta_{\pm} \mp \nu^{3/2}\frac{N_{\pm}^2}{\Gamma_{\pm}}\sum_{n=1}^{\infty} \mathrm{Re}([A_{\pm}^{(n)}|H|^{-(1/2)+i\mu_{\pm}}\nu^{-i\mu_{\pm}} + B_{\pm}^{(n)}|H|^{-(1/2)-i\mu_{\pm}}\nu^{i\mu_{\pm}}]e^{in\xi})\right]. \tag{22b}$$

The outer variables of the DBL follow from (22):

$$\Phi = \nu^2\varphi^{(0)},$$

$$\beta = \nu b^{(0)},$$

$$V = \nu u^{(0)} = \nu\ \frac{\partial\varphi^{(0)}}{\partial h},$$

$$X = \chi^{(0)} = \frac{\partial^2\varphi^{(0)}}{\partial h^2}.$$

The equations (17) are expressed in the outer variables in the following way:

$$\frac{\partial X}{\partial\xi}\frac{\partial\Phi}{\partial H} - \frac{\partial X}{\partial H}\frac{\partial\Phi}{\partial\xi} - Ri\ \frac{\partial\beta}{\partial\xi} = \lambda\nu^3\left(\frac{\partial^2 X}{\partial H^2} + 2\nu\ \frac{\partial X}{\partial\nu} + 2\left(H + \frac{s}{2} - \frac{\nu}{2}\frac{ds}{d\nu}\right)\frac{\partial X}{\partial H} + \frac{\alpha^2}{\nu^2}\frac{\partial^2 X}{\partial\xi^2}\right),$$

$$\frac{\partial\beta}{\partial\xi}\frac{\partial\Phi}{\partial H} - \frac{\partial\beta}{\partial H}\frac{\partial\Phi}{\partial\xi} = \frac{\lambda}{Pr}\ \nu^3\left(\frac{\partial^2\beta}{\partial H^2} + 2\nu\ \frac{\partial\beta}{\partial\nu} + 2\left(H + \frac{s}{2} - \frac{\nu}{2}\frac{ds}{d\nu}\right)\frac{\partial\beta}{\partial H} - 2\beta + \frac{\alpha^2}{\nu^2}\frac{\partial^2\beta}{\partial\xi^2}\right),$$

$$X = \frac{\partial^2\Phi}{\partial H^2} + \frac{\alpha^2}{\nu^2}\frac{\partial^2\Phi}{\partial\xi^2}.$$

The fields Φ, β and X can be presented as a sum of the averaged components and the ξ-dependent disturbances in the following way [see (22)]:

$$(\Phi,\beta,X) = (\Phi_0(H,\nu),\beta_0(H,\nu),X_0(H,\nu)) + \nu^{3/2}(\Phi_1(H,\nu),\beta_1(H,\nu),X_1(H,\nu))e^{i\xi}.$$

The solutions both for the averaged fields and for the disturbances should be posed as a series in ν. The linear approximation is valid and diffusion and time-dependence can be neglected for the disturbances of the fields in the zeroth order in ν. The system for disturbances is as follows:

$$\begin{aligned} &\Phi_{0H}X_1 - \Phi_1 X_{0H} - Ri\beta_1 = 0, \\ &\Phi_{0H}\beta_1 - \Phi_1\beta_{0H} = 0, \\ &X_1 = \Phi_{1HH} + \frac{\alpha^2}{\nu^2}\Phi_1. \end{aligned} \tag{23}$$

For the averaged fields diffusion and time-dependence should be taken into account (it is quite natural at the diffusion scale). The system of equations for the averaged fields in the lowest order in ν is as follows:

$$\frac{\partial^2 X_0}{\partial H^2} + 2\nu\ \frac{\partial X_0}{\partial\nu} + 2\left(H + \frac{s}{2} - \frac{\nu}{2}\frac{ds}{d\nu}\right)\frac{\partial X_0}{\partial H} = \frac{1}{\lambda}\frac{\partial^2 T}{\partial H^2}, \tag{24a}$$

$$\frac{\partial^2\beta_0}{\partial H^2} + 2\nu\ \frac{\partial\beta_0}{\partial\nu} + 2\left(H + \frac{s}{2} - \frac{\nu}{2}\frac{ds}{d\nu}\right)\frac{\partial\beta_0}{\partial H} - 2\beta_0 = \frac{Pr}{\lambda}\frac{\partial B}{\partial H}, \tag{24b}$$

$$X_0 = \frac{\partial^2\Phi_0}{\partial H^2}. \tag{24c}$$

Here T and B are the wave fluxes of momentum and mass, defined by the formula (10), (11), respectively. In the quasi-stationary approximation T and B do not depend on the vertical coordinate everywhere out of the small δ-vicinity of the CL [it follows from the systems (19) and (23)]. This scale in the variable H is equal to $\nu\delta \ll 1$. This means that in the zeroth order in ν there are delta-functions in the right hand of the equations (24a) and (24b), that are equivalent to definition of the boundary conditions on the CL.

The boundary condition on the CL for the outer solution can be obtained by matching outer and inner solutions on the CL. For the mean vorticity X_0 and velocity $V_0 = \partial\Phi_0/\partial H$ it follows from (22a), that

$$X_0(H,\nu) = \Gamma_{\pm}, \quad \text{for } H \gtrless 0, \tag{25a}$$

$$V_0(H,\nu) = \Gamma_{\pm}H + \nu c_{\pm}, \quad \text{for } H \gtrless 0. \tag{25b}$$

Taking (21a) into account yields

$$X_0(+0,\nu) - X_0(-0,\nu) = \Gamma_+ - \Gamma_- = \Xi(\nu), \tag{26}$$

where $\Xi = (T(+\infty) - T(-\infty))/\lambda$ is the normalized jump of the wave momentum flux across the CL.

The second and the third boundary conditions,

$$\begin{aligned} &V_0(+0,\nu) = 0, \\ &V_0(-0,\nu) = 0, \end{aligned} \tag{27}$$

follow from (25b) in the zeroth order in ν. The conditions (27) point out that in the 0-th order in $\nu\delta$ the velocity of the flow in the CL is equal to zero in the frame of reference moving with the wave phase velocity by definition.

The boundary condition on the CL for the density gradient can be obtained from (21b) similar to (26). Since the vertical mass flux in internal waves is equal to zero outside the CL, the boundary condition is as follows:

$$\frac{\partial \beta_0}{\partial H}(+0,\nu)=\frac{\partial \beta_0}{\partial H}(-0,\nu). \tag{28}$$

Taking into account (12) and the relation of η to H gives the boundary conditions for the velocity, vorticity and density at infinity in the following form:

$$V_0(\pm\infty,\nu)=H+\frac{s}{2}, \tag{29a}$$

$$X_0(\pm\infty,\nu)=1, \tag{29b}$$

$$\beta_0(\pm\infty,\nu)=-\left(H+\frac{s}{2}\right). \tag{29c}$$

It should be mentioned that (29c) obeys the equation (24b) and the boundary condition (28), i.e. it is valid for every H.

In the zeroth order in ν the source in the equation (24a) should be put equal to zero for $H>0$ and $H<0$, i.e. it takes the form

$$\frac{\partial^2 X_0}{\partial H^2}+2\nu\,\frac{\partial X_0}{\partial \nu}+2\left(H+\frac{s}{2}-\frac{\nu}{2}\,\frac{ds}{d\nu}\right)\frac{\partial X_0}{\partial H}=0. \tag{30}$$

The equation (30) with the boundary conditions (26), (27), (29a), (29b) enables us to study the evolution of the flow in the diffusion boundary layer at large time for an arbitrary, but sufficiently slow, dependence of the wave momentum flux on time.

IV. THE AVERAGE FIELDS OF VELOCITY AND VORTICITY IN THE DIFFUSION BOUNDARY LAYER. A CONSTANT JUMP OF THE WAVE MOMENTUM FLUX

Consider the most simple, but important case of a constant jump of the wave momentum flux across the CL, i.e. suppose that Ξ does not depend on ν. In this case, the equation (30) has a self-similar solution, the outer variables appear to be self-similar ones and the self-similar solution X_0 of (30) does not depend on ν (i.e., on time). Then X_0 obeys the ordinary differential equation of the second order,

$$\frac{d^2 X_0}{dH^2}+2\left(H+\frac{s}{2}\right)\frac{dX_0}{dH}=0. \tag{31}$$

The solution of (31) is as follows [the indexes ($\pm$) relate to $H>0$ and $H<0$]

$$X_0(H)=1+C_{\pm}\int_{\pm\infty}^{H+s/2} e^{-H_1^2}dH_1. \tag{32}$$

The boundary condition at infinity (29b) is taken into account in (32). Integrating of (32) with respect to H and taking into account the boundary condition (29a) yields

$$V_0(H)=H+\frac{s}{2}+C_{\pm}\left[\left(H+\frac{s}{2}\right)\int_{\pm\infty}^{H+s/2} e^{-H_1^2}dH_1+\frac{1}{2}\,e^{-(H+(s/2))^2}\right]. \tag{33}$$

The boundary conditions (26), (27) define the system of 3 equations for determining of three unknown values C_+, C_- and s as functions of Ξ which gives

$$C_-=-\frac{s}{e^{-s^2/4}+s(\sqrt{\pi}-J)},\qquad C_+=-\frac{s}{e^{-s^2/4}-sJ}, \tag{34}$$

here

$$J=\int_{s/2}^{\infty} e^{-H^2}dH.$$

The values of vorticity at both sides of the CL can be found from (32). Namely, up to the CL,

$$X(+0)=\Gamma_+=\frac{e^{-s^2/4}}{e^{-s^2/4}-sJ}, \tag{35a}$$

and down to the CL,

$$X(-0)=\Gamma_-=\frac{e^{-s^2/4}}{e^{-s^2/4}+s(\sqrt{\pi}-J)}. \tag{35b}$$

The equation determining s as a function of Ξ is as follows:

$$e^{-s^2/4}s\sqrt{\pi}=\Xi(e^{-s^2/4}-sJ)(e^{-s^2/4}+s(\sqrt{\pi}-J)).$$

The dependence $s(\Xi)$ is shown in Fig. 1 and the dependencies of $\Gamma_\pm$ on Ξ are presented in Fig. 2. A simple asymptotic expression can be found for $s(\Xi)$ and $\Gamma_\pm(\Xi)$ for large and small values of Ξ, so, for $\Xi \ll 1$,

$$s=\frac{\Xi}{\sqrt{\pi}},$$

$$\Gamma_\pm=1\pm\frac{\Xi}{2}.$$

The similar expression for vorticity on both sides of the CL was obtained,[6] for a Rossby wave on the β-plane, where the jump of vorticity related to the undisturbed one is the small value of order $\varepsilon^{1/2}$.

For $\Xi \gg 1$,

$$s=\sqrt{2\Xi},$$

$$\Gamma_+=\Xi,$$

$$\Gamma_-=\frac{e^{-\Xi/2}}{\sqrt{2\pi\Xi}}.$$

Substituting found values C_+ and C_- into the expressions (32) and (33) enables us to obtain the profiles of the

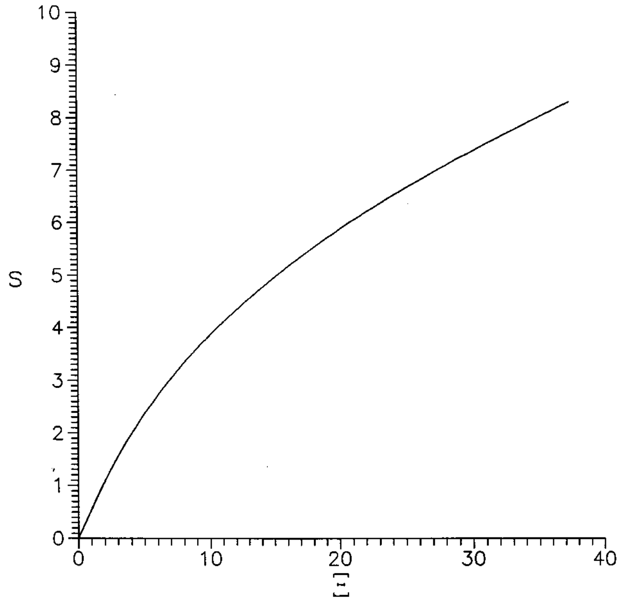

FIG. 1. The dependence of the constant in the rule of the CL motion (s) on the jump of vorticity (Ξ).

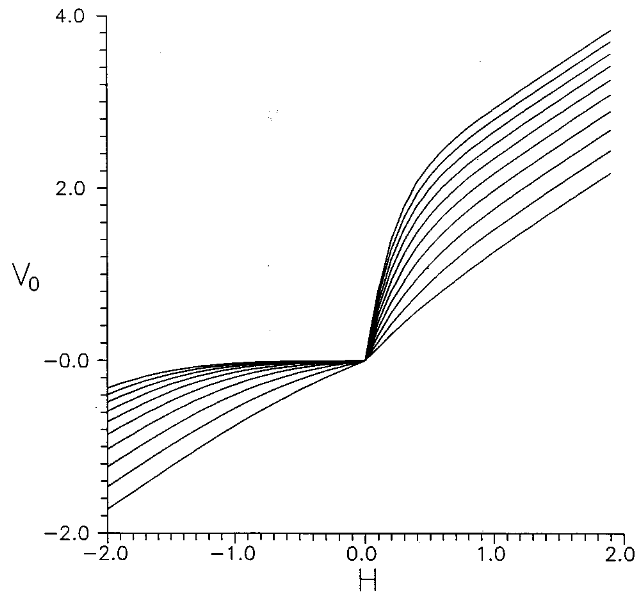

FIG. 3. The mean velocity profiles in the diffusion boundary layer. The jump of vorticity takes values from 1 (the lower curve) to 10 (the upper curve).

average velocity and vorticity in the DBL. The functions $V_0(H)$ are presented in Fig. 3 for several values of Ξ. The time evolution of the mean velocity profile $u_0(\eta,\tau)$ is shown in Fig. 4; it corresponds to the dependence $V_0(H)$ for one fixed Ξ. The velocity profile is widening with time and the CL is moving towards the incident wave momentum flux. Thus, for the simplest situation of the constant wave momentum flux we have determined the velocity profile in the DBL and found the values of mean vorticity on both sides of the CL for the known Ξ.

V. THE WAVE DISTURBANCE IN THE DIFFUSION BOUNDARY LAYER (CONSTANT WAVE MOMENTUM FLUX)

Consider now wave–flow interaction in the DBL and in the CL. The wave propagates from the region $H\rightarrow\infty$ (the outer region of the flow), where the velocity profile is deter-

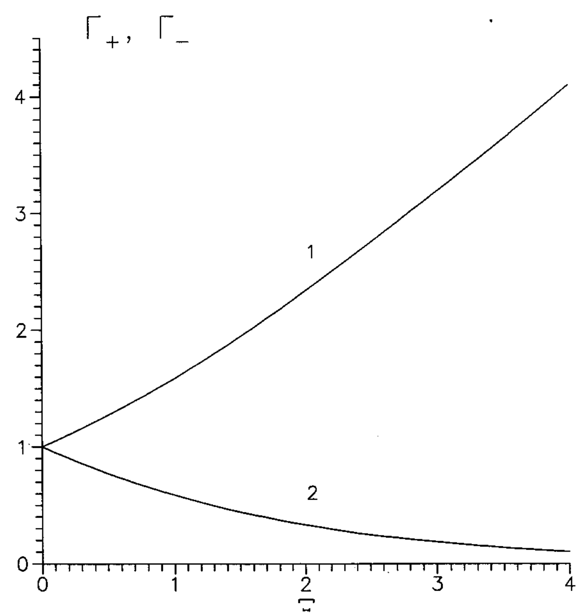

FIG. 2. The dependencies of the values of the mean vorticity at the different sides of the CL ($\Gamma_{\pm}$) on its jump (Ξ): 1. Γ_+—the value of the mean vorticity at the side of the incident wave; 2. Γ_-—the value of the mean vorticity at the side of the transmitted wave.

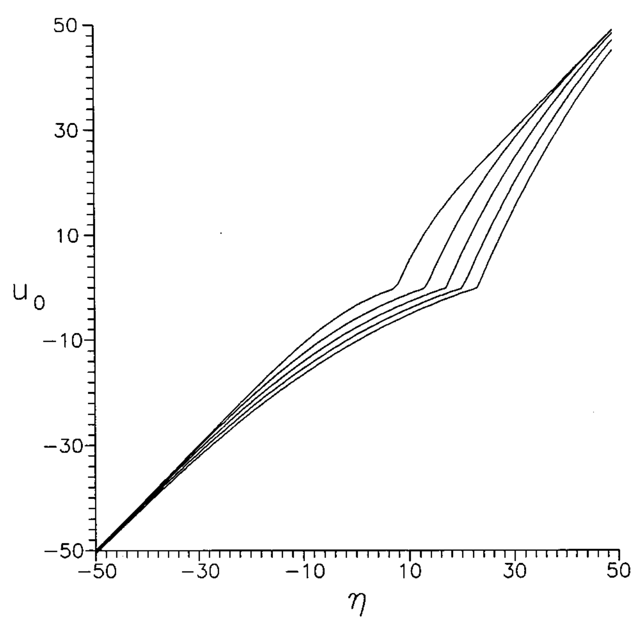

FIG. 4. The mean velocity profiles $u_0(\eta,\tau)$ in the diffusion boundary layer for τ=50, 100, 150, 200 (from left to right). The jump of vorticity Ξ=2.

mined by (29a). Propagating through the DBL, where the velocity profile is determined by (33), the wave partially reflects from the inhomogeneous vorticity profile, then the wave interacts with the CL partially reflecting, partially transmitting and partially absorbing. Note that the region of the flow, where the hydrodynamic fields have the asymptotic form (20a) and (20b), will be called the outer region of the CL; and the region, where the fields obey the nonlinear system (19) will be termed the inner region of the CL (or simply the CL). The inhomogeneous vorticity profile also occurs in the DBL down to the CL, so the transmitted wave scatters on it. Transmitted through the DBL the wave radiates down, and there is no incident wave radiated from minus infinity. As was mentioned above, the fields of the high harmonics are small in the DBL, so in this region we can take into account only the fundamental harmonic described by the system (23), which can be reduced to the Taylor–Goldstein equation,

$$\frac{d^2\Phi_1}{dH^2} - \frac{V_{0HH}}{V_0}\,\Phi_1 + \left(\frac{N^2}{V_0} - \frac{\alpha^2}{\nu^2}\right)\Phi_1 = 0. \tag{36}$$

The simplest case of $\alpha^2/\nu^2 \ll 1$ is considered here. When α^2/ν^2 is of order unity, nothing major would be changed, but the expressions of the fields would be more complicated.

The profile $V_0(H)$ tends to the undisturbed linear form (29a) for $H \to \pm\infty$. Taking that into account gives the asymptotic form of the field for $H \to \pm\infty$, i.e. the solution in the outer region,

$$\Phi_1(H\to\infty) = \mathscr{A}\nu^{(3/2)+i\mu}\left[\left(H+\frac{s}{2}\right)^{(1/2)-i\mu} + \mathscr{R}\left(H+\frac{s}{2}\right)^{(1/2)+i\mu}\right], \tag{37a}$$

$$\Phi_1(H\to-\infty) = \mathscr{A}\mathscr{T}\nu^{(3/2)+i\mu}\left[H+\frac{s}{2}\right]^{(1/2)-i\mu}, \tag{37b}$$

where $\mu = \sqrt{Ri - 1/4}$. The factor $\nu^{(3/2)+i\mu}$ arises due to normalization. The formula (37b) is the radiation condition for $H \to -\infty$; it means that no wave propagates from the inner region of the flow down to the CL.

The asymptotic expressions (20a) valid in the outer region of the CL are the fields on the ''output'' of the DBL, on the ''input'' of which the asymptotics (37a) and (37b) are set. The amplitudes and reflection coefficients of the waves on the ''input'' and the ''output'' are connected by the solution of the Taylor–Goldstein equation (36)

The expressions connecting $\mathscr{A}$ and $\mathscr{R}$ (37a) with $A_+^{(1)}$ and $B_+^{(1)}$ from (21a) and $\mathscr{T}$ (37b) with $A_-^{(1)}$ and $B_-^{(1)}$, are found below. Introduce the symbols $A = A_+^{(1)}$ (the amplitude of the wave, propagating towards the CL from its outer region) and R (the reflection coefficient of the unit amplitude wave from CL). Then, as was shown in **I**, $B_+^{(1)} = AR|A|^{-(4/3)i\mu_+}$. And for $H \ll 1$ the expression for the stream function disturbance is as follows:

$$\Phi_1 = A\nu^{(3/2)+i\mu}\left[H^{(1/2)-i\mu_+} + RH^{(1/2)+i\mu_+}|A|^{-(4/3)i\mu_+}\nu^{-2i\mu_+}\right]. \tag{38}$$

Passing through the DBL transforms the wave field (38) into (37a).

Suppose that the initial condition for the equation (36) for $H \ll 1$ are imposed in the form of a wave propagating towards the CL, i.e.

$$\Phi_{0-} = H^{(1/2)-i\mu_+};\quad \frac{d\Phi_{0-}}{dH} = \left(\frac{1}{2} - i\mu_+\right)H^{-(1/2)-i\mu_+}. \tag{39a}$$

For $H \gg 1$ the solution to (36) can be presented in the following way:

$$\Phi_{\infty-} = a\nu^{(3/2)+i\mu}\left[\left(H+\frac{s}{2}\right)^{(1/2)-i\mu} + r\left(H+\frac{s}{2}\right)^{(1/2)+i\mu}\right],$$

where a and r are the complex constants, depending on the parameters of the equation (36). r is the coefficient of reflection from the DBL, a is the amplitude of the wave propagating towards the DBL for the unity amplitude of the transmitted wave. If the initial conditions for $H \ll 1$ are imposed as the wave propagating from the CL:

$$\Phi_{0+} = H^{(1/2)+i\mu_+};\quad \frac{d\Phi_{0+}}{dH} = \left(\frac{1}{2} + i\mu_+\right)H^{-(1/2)+i\mu_+}; \tag{39b}$$

then for $H \gg 1$ the wave field can be written in the following way:

$$\Phi_{\infty+} = \Phi_{\infty-}^* = a^*\nu^{(3/2)-i\mu}\left[\left(H+\frac{s}{2}\right)^{(1/2)+i\mu} + r^*\left(H+\frac{s}{2}\right)^{(1/2)-i\mu}\right],$$

where ( )* denotes complex conjugation. It obviously follows from linearity of the equation (36), absence of the critical point on the interval considered and the fact that $\Phi_{0+} = \Phi_{0-}^*$.

If the initial conditions for (36) are formulated in the form (38), then for $H \gg 1$ the solution is a linear combination of Φ_{0+} and Φ_{0-}, namely,

$$\Phi_\infty = A\nu^{(3/2)+i\mu}\left[\Phi_{0-} + R\Phi_{0+}|A|^{-(4/3)i\mu_+}\nu^{-2i\mu_+}\right]. \tag{40}$$

On the other hand, (40) must coincide with (37a). Comparing (37a) and (40) gives

$$\mathscr{A} = |a|Ae^{i\varphi_a}\left(1 + |R||r|e^{-i(2\varphi_a+\varphi_r-\varphi_R)}\times|A|^{-(4/3)i\mu_+}\nu^{-2i\mu_+}\right), \tag{41}$$

$$\mathscr{R} = \nu^{2i\mu_+}e^{i\varphi_r}\,\frac{|r| + |R|e^{-i(2\varphi_a+\varphi_r-\varphi_R)}|A|^{-(4/3)i\mu_+}\nu^{-2i\mu_+}}{1 + |R||r|e^{-i(2\varphi_a+\varphi_r-\varphi_R)}|A|^{-(4/3)i\mu_+}\nu^{-2i\mu_+}}. \tag{42}$$

Here $|a|$, φ_a; $|r|$, φ_r and $|R|, \varphi_R$ are the modulus and arguments of the complex values a, r and R. The complex values r and a can be found by solving the equation (36) with the initial conditions (39a). The dependence of r, a, φ_r and φ_a on Ξ for the set of the Richardson numbers Ri are presented in Figs. 5, 6, 7, 8.

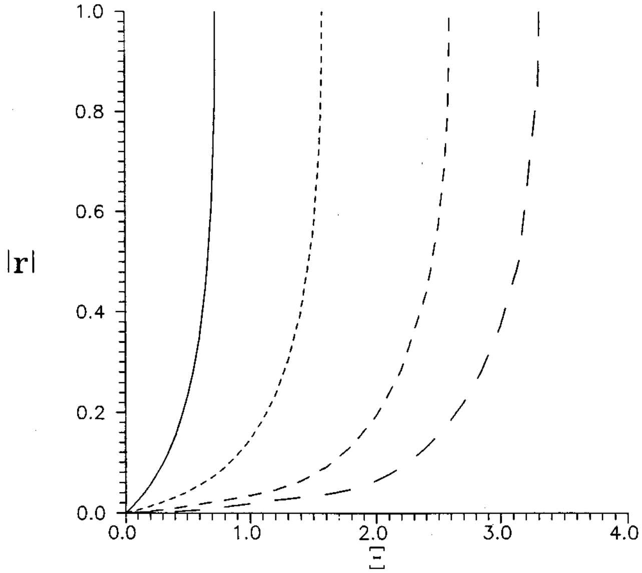

FIG. 5. The dependence of $|r|$ on Ξ ——$Ri=0.5$, --- $Ri=1$, ––– $Ri=2$, —— $Ri=3$.

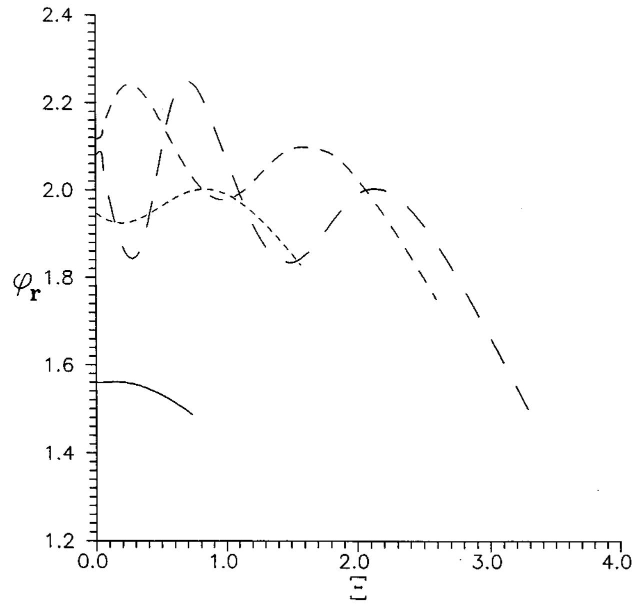

FIG. 7. The dependence of φ_r on Ξ ——$Ri=0.5$, --- $Ri=1$, ––– $Ri=2$, — -$Ri=3$.

Solving the equation (36) with the initial conditions (37b) for $H\rightarrow-\infty$ (with the corresponding first derivative) gives the expressions for $A_{-}^{(1)}$ and $B_{-}^{(1)}$ from (22a) and (22b) by $\mathscr{T}$ from (37b).

It was assumed above, that the vorticity jump across the CL does not depend on time, and so the amplitude of the incident wave A and the coefficient of reflection R from the CL in the outer region of the CL (not in the outer region of the flow!) do not depend on time. More strictly, if the radiation condition for $H\rightarrow-\infty$ is valid, then the wave field in the lower outer region depends on time, because the wave transmitted through the CL scatters on the time dependent DBL down to the CL. But it slightly affects the dynamics of the CL, because the transmitted wave amplitude is very small for moderate and large Richardson numbers (see **I**); it is of order $e^{\mu\pi}$. So the wave field up to the CL can only very slightly depend on time, and A and R are practically constant for the wave momentum flux jump (Ξ). Then for the constant Ξ, it follows from (41), (42) that in the outer region of the flow the amplitude of the incident wave $\mathscr{A}$ and the wave reflection coefficient $\mathscr{R}$ depend on time. This time dependence arises from the reflection of the incident wave from the time-dependent velocity profile in the DBL (widening and mov-

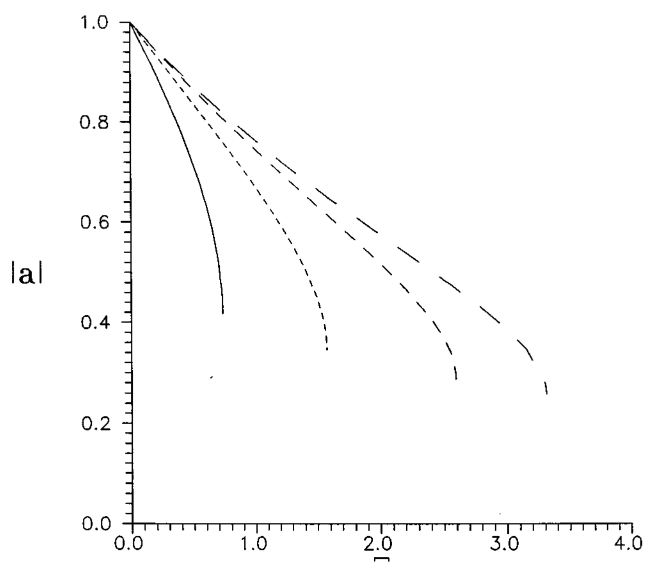

FIG. 6. The dependence of $|a|$ on Ξ ——$Ri=0.5$, ---$Ri=1$, ––– $Ri=2$, — $Ri=3$.

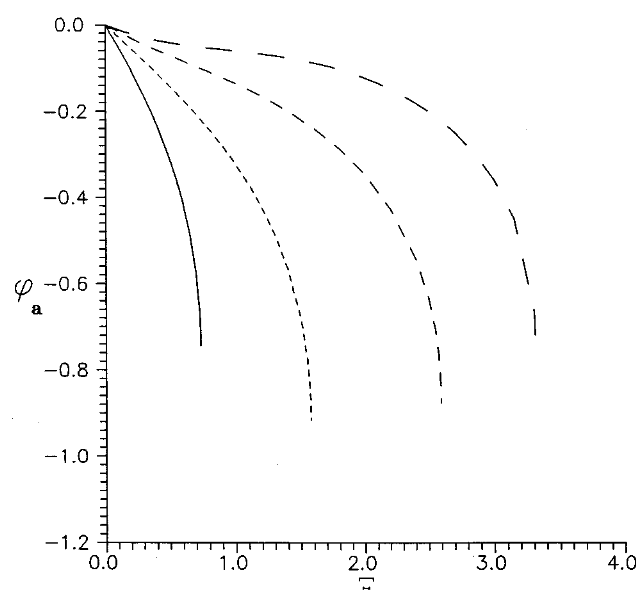

FIG. 8. The dependence of φ_a on Ξ ——$Ri=0.5$, ---$Ri=1$, ––– $Ri=2$, —— $Ri=3$.

ing). To provide conservation in time of the sum wave momentum flux in the incident and reflected waves the amplitude $\mathscr{A}$ and the reflection coefficient $\mathscr{R}$ should depend on time. Alternatively, the source of the wave disturbance placed into the outer region of the flow should depend on time according to a certain law. On the other hand, if the source placed in the outer region of the flow is, for example, time independent, then the jump of the wave momentum flux depends on time. This case is discussed below in section VII.

VI. THE CHARACTERISTICS OF THE CL

To find the parameters A and R in (41), (42) the system (19) for the fields in the CL vicinity should be solved. But before that the expression (21a) connecting the amplitudes of the incident, reflected and transmitted waves with the vorticity jump across the CL will be rewritten in terms of $\mathscr{T}$—the coefficient of transmission of the wave through the DBL down to the CL. Taking into account conservation of the wave momentum flux in the waves down to the CL yields

$$\mu_-(|B_-^{(1)}|^2-|A_-^{(1)}|^2)=\mu|\mathscr{T}|^2. \tag{43}$$

And (21a) together with (43) gives

$$\lambda(\Gamma_+-\Gamma_-)=\mu_+|A|^2(1-|R|^2)-\mu|\mathscr{T}|^2. \tag{44}$$

The system (19) was solved numerically using a spectral model which was described in detail in **I**. Afterwards all the characteristics of the flow and waves can be found as functions of the parameters of the flow, namely, the parameter of nonlinearity λ, the Richardson number Ri and the Prandtl number Pr. The parameter λ was determined by the amplitude A of the incident wave in the outer region of the CL, because for the simple case of the constant jump of the wave momentum flux, $\mathscr{A}$ depends on time, but A does not [see (41)]. In this case the inner variables are renormalized in the following way:

$$h=\frac{h_{old}}{|A|^{2/3}};\quad \varphi=\frac{\varphi_{old}}{|A|^{4/3}};\quad b=\frac{b_{old}}{|A|^{2/3}},$$

and λ is the following:

$$\lambda=\frac{\lambda^{(old)}}{|A|^2}.$$

Taking that into account enables us to represent the flux equation (44) in the form

$$\lambda(\Gamma_+-\Gamma_-)=\mu_+(1-|R|^2)-\mu|Tr|^2, \tag{45}$$

here $Tr=\mathscr{T}/|A|$. It should be mentioned that there is $\lambda^{(old)}$ in (44) and λ in (45).

The dependencies on λ of the values appearing in (45) for a set of Richardson numbers are represented in Figs. 9–13, namely $\Xi(\lambda)$—in Fig. 9, $\Gamma_+(\lambda)$—in Fig. 10, $\Gamma_-(\lambda)$—in Fig. 11, $|R|(\lambda)$—in Fig. 12 and $|Tr|e^{\pi\mu}(\lambda)$—in Fig. 13. Besides, the dependence of $\varphi_R(\lambda)$ which is needed for calculation of the values $\mathscr{A}$ and $\mathscr{R}$ by (41) and (42) is presented in Fig. 14. It should be mentioned that the dependencies in Figs. 9–14 differ from the similar ones from **I**, which were obtained for the constant Richardson number in the outer region of the CL down to the CL (because of the normalization of the vorticity by its value in this region). In the present paper each curve in Figs. 9–14 corresponds to the constant Richardson number in the undisturbed region of the flow. As in the paper **I** the simple formula for $\lambda(\Xi)$ which is valid for small $|R|$ and $|Tr|$ can be obtained from (45). It is as follows:

$$\lambda=\frac{\sqrt{\dfrac{Ri}{\Gamma_+^2}-\dfrac{1}{4}}}{2\Xi}, \tag{46}$$

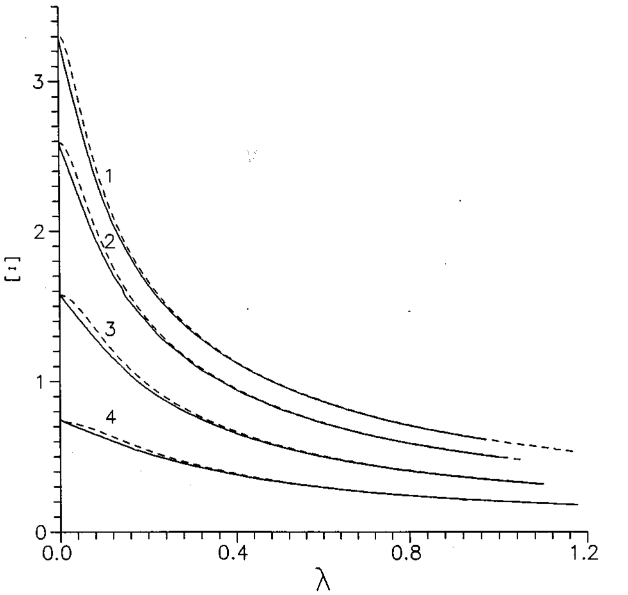

FIG. 9. The dependence of the jump of the mean vorticity across the CL Ξ on λ for the following values of the Richardson number: 1. $Ri=0.5$, 2. $Ri=1$, 3. $Ri=2$, 4. $Ri=3$. Prandtl number $Pr=0.71$. Dashed curves represent the function $\Xi(\lambda)$ calculated by (46).

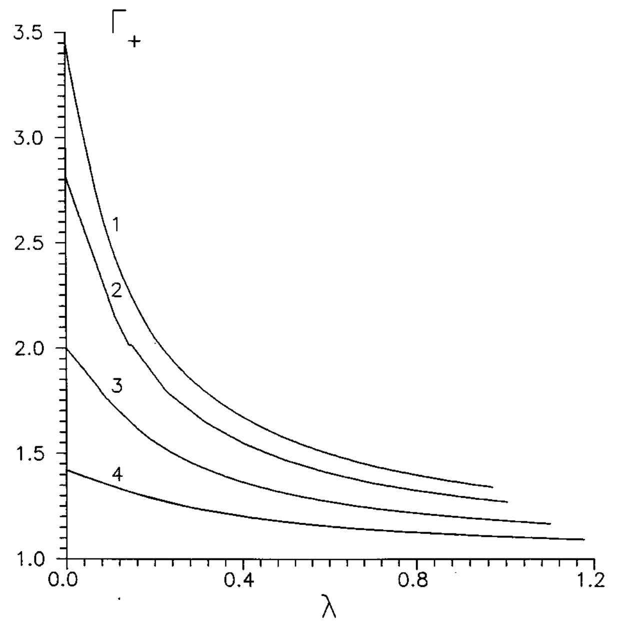

FIG. 10. The dependence of the vorticity value above the CL Γ_+ on λ for the following values of the Richardson number: 1. $Ri=0.5$, 2. $Ri=1$, 3. $Ri=2$, 4. $Ri=3$. Prandtl number $Pr=0.71$.

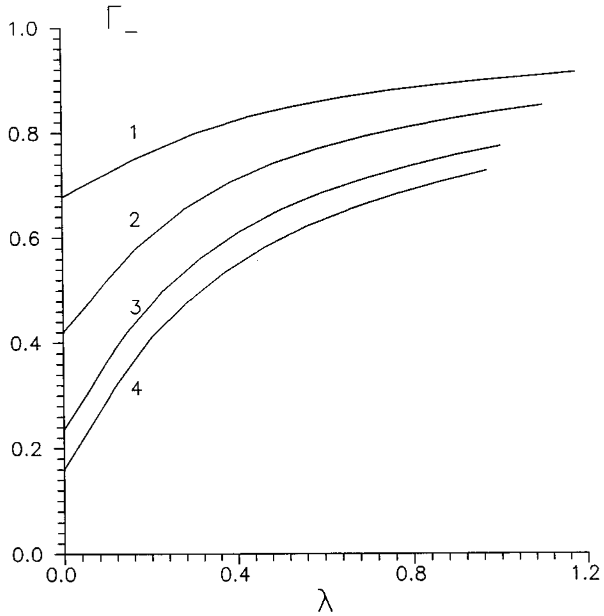

FIG. 11. The dependence of the vorticity value below the CL Γ_- on λ for the following values of the Richardson number: 1. $Ri=0.5$, 2. $Ri=1$, 3. $Ri=2$, 4. $Ri=3$. Prandtl number $Pr=0.71$.

where Γ_+ is determined by (35a). The dashed curves in fig. 9 correspond to the relation (46). They are close to the curve obtained numerically.

VII. THE TIME-DEPENDENT JUMP OF THE WAVE MOMENTUM FLUX

All the previous results concern the case of a constant wave momentum flux propagating towards the CL. In this case, the evolution of the mean fields in the DBL is self-similar. And the characteristics of the fields (namely, the mean velocity profile, the law of the CL displacement, the values of the mean vorticity up and down the CL) in this region can be obtained easily. In this case the amplitude of the incident wave A and the reflection coefficient R in the outer region of the CL are constant in time. But in the outer region the incident wave amplitude $\mathcal{A}$ and the reflection coefficient $\mathcal{R}$ depend on time [see (41), (42)]. On the other hand, if the source of the wave disturbance is more realistic

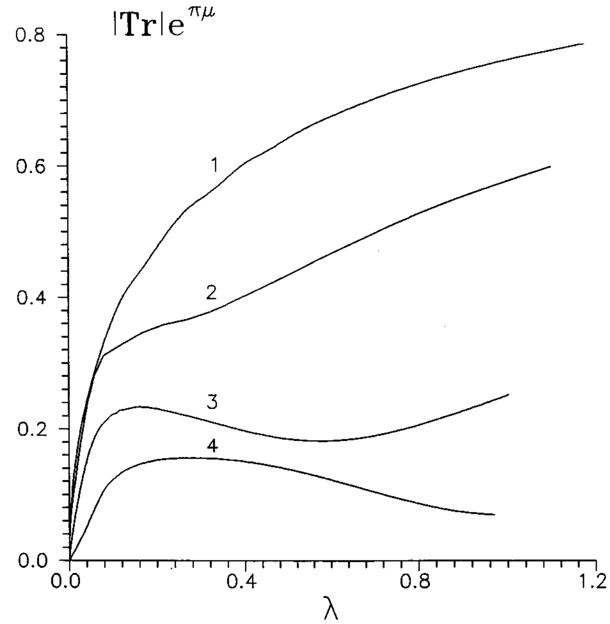

FIG. 13. The dependence of the normalized value of the transmission coefficient $|Tr|e^{\pi\mu}$ on λ 1. $Ri=0.5$, 2. $Ri=1$, 3. $Ri=2$, 4. $Ri=3$. Prandtl number $Pr=0.71$.

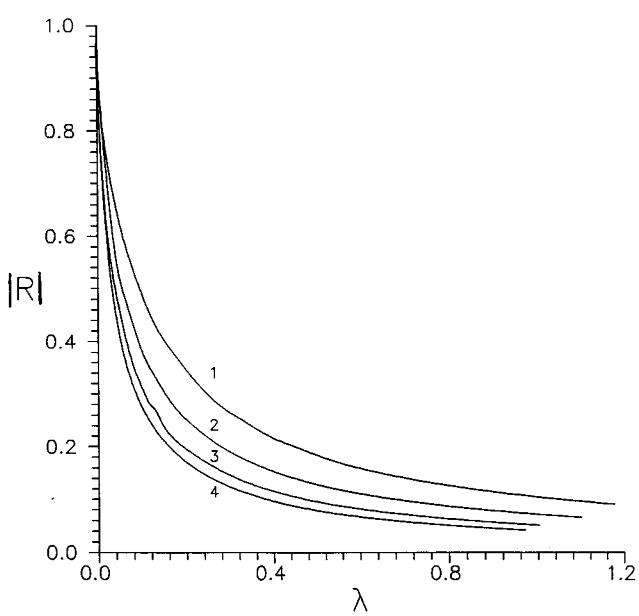

FIG. 12. The dependence of the absolute value of the reflection coefficient $|R|$ on λ for the following values of the Richardson number: 1. $Ri=0.5$, 2. $Ri=1$, 3. $Ri=2$, 4. $Ri=3$. Prandtl number $Pr=0.71$.

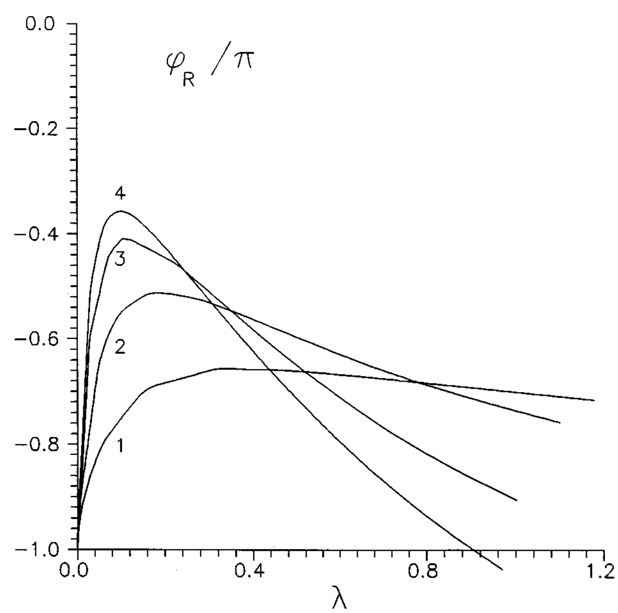

FIG. 14. The dependence of the phase of the reflection coefficient φ_R on λ for the following values of the Richardson number: 1. $Ri=0.5$, 2. $Ri=1$, 3. $Ri=2$, 4. $Ri=3$. Prandtl number $Pr=0.71$.

(for example the corrugated surface in the outer region of the flow) the values of the wave momentum fluxes on both sides of the CL and the jump of those will depend on time. The origin of that is as follows. The wave, radiating by the source in the outer region of the flow, propagates towards the DBL, partially reflects at the inhomogeneous profile of vorticity in the DBL and partially transmits. Then the transmitted wave interacts with the DBL behind the CL. Since the velocity profile in the DBL depends on time (because it is spreading) the reflection coefficient depends on time as well. It means that the momentum flux, which is equal to the sum of fluxes in the incident and reflected waves, depends on time. In this case the simple self-similar velocity profile can not be obtained. And the full equation (30) with the boundary conditions (26), (27), (29a), (29b), together with the equation (36) for the wave disturbances in the DBL, should be studied. It is impossible to find the general solution to the problem, but the approximate solution can be obtained for the certain situations.

First, if the Richardson number is not very close to 1/4, the transmitted wave behind the CL is very small, so the wave momentum flux in the transmitted wave $[T(-\infty)]$ can be neglected and the jump of the wave momentum flux is approximately equal to $T(+\infty)$, and $\Xi = T(+\infty)/\lambda$. Second, since the time-dependence of the wave momentum flux arises due to the reflection of the wave from the time-dependent velocity profile, one can expect that the time-dependent component of the wave momentum flux is small when the reflection coefficient is small. As one can see from Fig. 12 and from comparison of Fig. 5 and Fig. 9, that $|R|,|r| \leqslant 0.1$ for the moderate values of the Richardson number (not very close to 1/4) and $\lambda > 0.2$. Then the solution of the equation can be searched as a series in $\mathscr{R}$.

Consider in this way the flow over a corrugated surface when the reflection coefficients R and r (and $\mathscr{R}$) are small. The undisturbed velocity profile of the flow in the dimensional variables is as follows:

$$U_d(z_d) = u_{0z} z_d .$$

The corrugated surface placed at $z_d = z_F$ has the form

$$z_d = z_F + h_0 \operatorname{Re} e^{i k_d x_d}, \tag{47}$$

here k_d is the dimensional wave number.

The stream function of this flow outside the DBL is as follows:

$$\Psi = \frac{u_{0z} z_d^2}{2} + \operatorname{Re}\left\{ \Psi_0 e^{i k_d x_d} \left[\left(\frac{z_d}{z_F}\right)^{(1/2)-i\mu} + \tilde{\mathscr{R}} \left(\frac{z_d}{z_F}\right)^{(1/2)+i\mu} \right] \right\}. \tag{48}$$

Here $\tilde{\mathscr{R}}$ is the reflection coefficient, the absolute value of $\tilde{\mathscr{R}}$ coincides with that of $\mathscr{R}$ (42) and the argument is determined by the normalization condition.

The natural scaling values of this problem (see sec. II) are $L_0 = z_F$, $U_0 = u_{0z} z_F$, then the dimensionless variables become $z = z_d/z_F$, $x = x_d/z_F$, $k = k_d z_F$, $\psi = \Psi/(u_{0z} z_F^2)$, etc. The Reynolds number $Re = (u_{0z} z_F^2)/\nu_0$. Taking into account this scaling and the expression (48) for Ψ and neglecting the transmitted wave for $Ri \geqslant 1$ gives the following expression for the jump of the wave momentum flux, expressed by the dimensional parameters of the flow and the surface:

$$\Xi = \frac{k_d \mu |\Psi_0|^2 (1 - |\mathscr{R}|^2)}{2 \nu_0 u_{0z} z_F}. \tag{49}$$

In the zeroth order in $\mathscr{R}$ it follows from (47) and (48) that $\Psi_0 = u_{0z} z_F h_0$ and (49) gives

$$\Xi^{(0)} = \frac{1}{2\nu_0} k_d \mu u_{0z} z_F h_0^2.$$

The jump of the wave momentum flux $\Xi^{(0)}$ is constant and the velocity profile in the DBL and all the characteristics of the flow in the zeroth order can be obtained from Figs. 1–14 by Ri and $\Xi^{(0)}$, namely $s^{(0)}, \lambda^{(0)}, \Gamma_{\pm}^{(0)}, R^{(0)}, r^{(0)}, a^{(0)}$. The index $^{(0)}$ is omitted below. Then the reflection coefficient of the wave in the outer region of the flow is

$$\mathscr{R} = \nu^{2i\mu} e^{i\varphi_r} (|r| + R e^{-i(2\varphi_a + \varphi_r)} \nu^{-2i\mu_+}). \tag{50}$$

It obviously follows from (50) that $\mathscr{R}$ depends on time as the sum of two $\cos(\log \tau)$, where ν is related to τ by (16). Calculating all the parameters in the zeroth order in $\mathscr{R}$ enables to find the first order term of Ξ. Comparing (47) and (48) gives in the first order in $\mathscr{R}$, that $\Psi_0 = u_{0z} z_F h_0 (1 - \tilde{\mathscr{R}})$ Taking into account $\tilde{\mathscr{R}} = \mathscr{R} e^{-2i\mu \ln \Delta}$ [where $\Delta = (h_0/Z_F)^{2/3} (\mu_+/\mu)^{1/3}$] and the normalizing conditions gives

$$\Xi = \frac{1}{2\nu_0} k_d \mu u_{0z} h_0^2 z_F [1 - 2 \operatorname{Re}(r_1 \nu^{2i\mu_1} + r_2 \nu^{2i\mu_2})].$$

Here $r_1 = |r| e^{i(\varphi_r - 2\mu \ln \Delta)}$, $r_2 = |R| e^{i(\varphi_a - 2\mu \ln \Delta)}$, $\mu_1 = \mu$, $\mu_2 = \mu - \mu_+$. So the jump of the wave momentum flux across the CL is the sum of the constant component and the term, oscillating in time as the sum of two $\cos(\log \tau)$.

The mean vorticity χ can be searched in the similar way, namely as a sum of the mean and oscillating terms,

$$\chi = \chi_0(H) - 2 \operatorname{Re}(\chi_1(H) \nu^{2i\mu_1} + \chi_2(H) \nu^{2i\mu_2}). \tag{51}$$

The parameter $s(\tau)$ in the law of the CL displacement (15) can be determined in the same way:

$$s = s^{(0)} - 2 \operatorname{Re}(s_1 \nu^{2i\mu_1} + s_2 \nu^{2i\mu_2}).$$

Then χ_1 and χ_2 obey the following equations:

$$\frac{\partial^2 \chi_{1.2}}{\partial H^2} + 2\left(H + \frac{s^{(0)}}{2}\right) \frac{\partial \chi_{1.2}}{\partial H} + 4 i \mu_{1,2} = C_{\pm} e^{-[H + (s^{(0)}/2)]^2} s_{1,2} (1 - 2 i \mu_{1,2}); \tag{52}$$

here $C_{\pm}$ are given by the formula (34). The solution to the equation (52) has the following form:

$$\chi_{1,2}(H) = \left[\frac{C_{\pm}}{2} s_{1,2} - B_{\pm}^{(1,2)} D_{-1+2i\mu_{1,2}} \times \left(\pm \left(H + \frac{s^{(0)}}{2} \right) \sqrt{2} \right) \right] e^{-[H + (s^{(0)}/2)]^2}. \tag{53}$$

Here $D_{-1+2i\mu_{1,2}}(Q)$ is the solution to the Weber equation,

$$\frac{d^2 D_n}{dQ^2} + \left(n + \frac{1}{2} - \frac{Q^2}{4} \right) D_n = 0,$$

vanishing for $Q \to \infty$.

The expressions for the mean velocity $V_{1,2}(H)$ vanishing for $H \to \infty$, which correspond to (53) are the following:

$$V_{1,2} = \frac{s_{1,2}}{2} + \frac{C_{\pm}}{2} s_{1,2} \int_{\pm\infty}^{H+(s^{(0)}/2)} e^{-H_1^2} dH_1 + B_{\pm}^{(1,2)} \int_{\pm\infty}^{H+(s^{(0)}/2)} D_{-1+2i\mu_{1,2}}(\pm H_1 \sqrt{2}) e^{-H_1^2} dH_1 .$$

The values $s_{1,2}, B_{\pm}^{(1,2)}$ are determined by the boundary conditions:

$$\chi_{1,2}(+0) - \chi_{1,2}(-0) = \Xi^{(0)} r_{1,2}, \quad V_{1,2}(\pm 0) = 0. \tag{54}$$

Two systems of three equations for determining two sets of three complex values $s_1, B_{\pm}^{(1)}$ and $s_2, B\pm^{(2)}$ follow from the boundary conditions (54). It obviously follows from (51) that the values of vorticity on both sides of the CL $(\Gamma_{\pm})$ can be represented as the sum of the constant $(\Gamma_{\pm}^{(0)})$ and the oscillating components with the amplitudes $(\Gamma_{\pm}^{(1,2)})$:

$$\Gamma_{\pm} = \Gamma_{\pm}^{(0)} - 2\,\mathrm{Re}(\Gamma_{\pm}^{(1)} \cdot \nu^{2i\mu} + \Gamma_{\pm}^{(2)} \nu^{2i(\mu-\mu_+)}).$$

The complex values $s_{1,2}$, $\Gamma_{\pm}^{(1,2)}$ are functions of two parameters—$\Xi^{(0)}$, Ri (or $\lambda^{(0)}$, Ri taking into account the dependence $\Xi^{(0)}(\lambda^{(0)})$ presented in Fig. 9). The example of dependence $\Gamma_{\pm}^{(1,2)}(\Xi^{(0)})$ is shown in Figs. 15a,b for $Ri=2$. The linear approximation in $\mathscr{R}$ is obviously valid for $\Xi^{(0)} < 1.5$ or as it follows from Fig. 9 for $\lambda > 0.2$.

VIII. CONCLUSION

In the present paper a stratified shear flow over a corrugated surface is considered. The velocity of the flow changes its direction at some level z_0 over the surface. Then the critical layer (CL) is formed in the vicinity of this level, where the flow velocity coincides with the phase velocity of the lee waves (which is equal to zero in this problem). The corrugated surface is supposed to have the simple sinusoidal shape, i.e. the elevation of the surface is

$$h = h_0 \cos k_d x_d$$

(x is the horizontal coordinate). If the amplitude h_0 is small enough, that the parameter $\varepsilon = h_0/z_0 \ll 1$, and the Reynolds number $Re = U_0/(\nu_0 k)$ is large enough, then the linear inviscid approximation can be used for wave far from the CL. And the complex amplitude $\psi(z)$ of the stream function disturbance obeys the Taylor–Goldstein equation with singularities in the CL. So $\psi(z)$ have the algebraic branch points in the CL. For example, for the wave propagating towards the CL,

$$\psi = (z_d - z_0)^{(1/2) - i\mu}, \tag{55}$$

where $\mu = \sqrt{Ri - 1/4}$.

To remove singularities some additional factors should be taken into account: viscosity, nonlinearity and nonstationarity.[13–15] In the present paper we concentrate on the combined effect of nonlinearity and dissipation in the stationary CL forming after the large time from the beginning of the process of wave–flow interaction. The main properties of this kind of the CL in the strongly stratified shear flows are obtained in **I**. The brief description of them is given below.

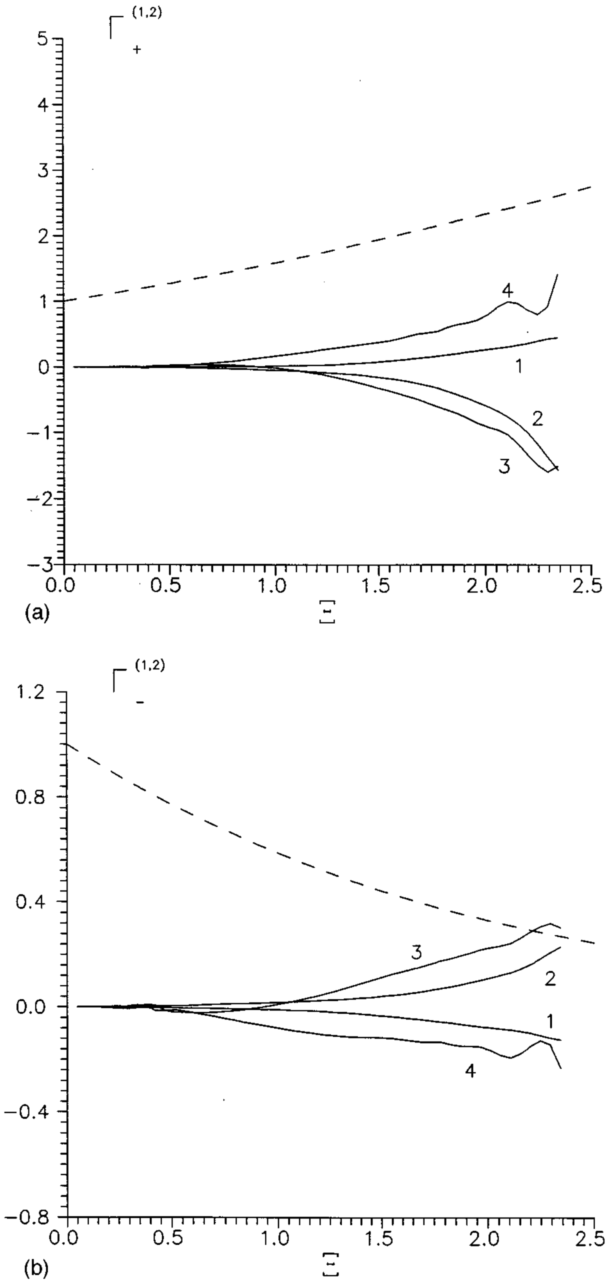

FIG. 15. (a) Complex amplitudes of the oscillating components of the mean vorticity up to the CL—$\Gamma_{+}^{(1,2)}$, $Ri=2$. 1. $\mathrm{Re}(\Gamma_{+}^{(1)})$, 2. $\mathrm{Im}(\Gamma_{+}^{(1)})$, 3. $\mathrm{Re}(\Gamma_{+}^{(2)})$, 4. $\mathrm{Im}(\Gamma_{+}^{(2)})$. The dashed curve is $\Gamma_{+}^{(0)}(\Xi)$. (b) Complex amplitudes of the oscillating components of the mean vorticity down to the CL—$\Gamma_{-}^{(1,2)}$, $Ri=2$. 1. $\mathrm{Re}(\Gamma_{-}^{(1)})$, 2. $\mathrm{Im}(\Gamma_{-}^{(1)})$, 3. $\mathrm{Re}(\Gamma_{-}^{(2)})$, 4. $\mathrm{Im}(\Gamma_{-}^{(2)})$. The dashed curve is $\Gamma_{-}^{(0)}(\Xi)$.

The jump of mean vorticity (Fig. 16a) or ''bending'' of the velocity profile (Fig. 16b) appears across the CL. When $Ri > 1/4$ the jump appears in the 0-th order of ε. The width of the jump is equal to the width of the CL (proportional to $\varepsilon^{2/3}$). The value of the vorticity at the side of incident wave $(dU_d/dz_d|_{+})$ is larger than that at the other side $(dU_d/dz_d|_{-})$, and the more the amplitude of the incident wave the larger the vorticity jump $(dU_d/dz_d|_{+} - dU_d/dz_d|_{-})$. The jump of vorticity is deter-

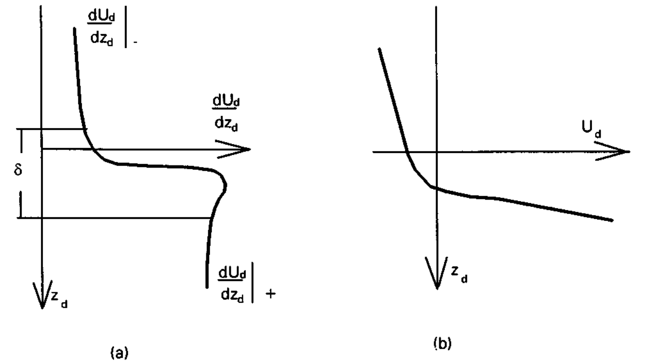

FIG. 16. The forms of the mean profiles of vorticity (a) and vorticity (b) in the vicinity of the CL.

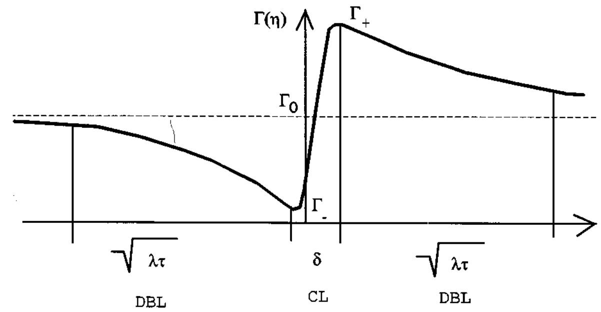

FIG. 17. The form of the mean vorticity profile in CL and DBL.

mined by the jump of the wave momentum flux at the both sides of the CL. In the dimensional form the relating expression is as follows:

$$\nu_0\left(\left.\frac{dU_d}{dz_d}\right|_+ - \left.\frac{dU_d}{dz_d}\right|_-\right) = \langle u_{d\sim} w_{d\sim}\rangle|_+ - \langle u_{d\sim} w_{d\sim}\rangle|_-, \qquad (56)$$

where ν_0 is the viscosity coefficient, $u_{d\sim}$, $w_{d\sim}$ wave disturbances of the horizontal and vertical velocities, and $\langle\cdots\rangle$ means averaging over the wave disturbances. In the dimensionless form this expression is given by (21a). It reflects the fact that in the stationary CL the radiation force is equal to the viscous force. Another property of the dissipation nonlinear CL is the reflected wave. The well known fact is, that the wave of infinitesimal amplitude does not reflect from stratified shear flow with the homogeneous basic vorticity profile [see (55)]. Alternatively the wave of small but finite amplitude reflects from the inhomogeneous profile of the mean vorticity with the jump across the CL. The jump of vorticity and the complex reflection coefficient are the functions of the parameters of the flow in the CL are the following: the Richardson number Ri—and the nonlinearity parameter $\lambda = 1/(\varepsilon^2 Re)$ (λ is the inverse inner Reynolds number of the flow in the CL vicinity).

The jump of vorticity (or ''bending'' of the velocity profile) in the CL vicinity means that the deformation of the initial velocity profile grows with the distance from the CL. The question arises, how this velocity profile can be realized. The same question concerning the flow velocity deformation arose in the works by Haberman.[1,2] The other question is as follows. If the initial velocity profile, the stratification and the amplitude of the incident wave are known, what the values of the vorticity at the both sides of the CL after the process of relaxation will be. To answer these questions the initial problem of the internal wave propagation towards the CL in the stratified shear flow is solving. The average horizontal velocity u_0 obeys the diffusion equation (9), which demonstrates that the average acceleration of the fluid particle is determined by the friction force and the radiation force. The equation is accompanied by the boundary and the initial conditions of the absence of disturbances of the velocity profile in the starting moment and at infinity. The average density obeys the similar equation and the boundary and initial conditions.

The evolution of the flow after large time from the beginning of the process in comparison with the diffusion time at the scale of the CL (13) is determined by the qualitative properties of the equation (9). At that time the flow in the vicinity of the CL is quasi-stationary. Since the diffusion is decelerating process the CL becomes ''more stationary'' with time. The jump of vorticity determined by (56) appears like in the stationary CL. The value of the vorticity determined at the side of the wave incident is larger than the undisturbed one, at the other side it is smaller than undisturbed one. CL appears to be the kind of source of the mean vorticity and the diffusion of vorticity from the CL occurs due to viscosity, so the transitional region from the CL to the undisturbed region is forming (Fig. 17). This region is called the diffusion boundary layer (DBL). Its scale is the diffusion length $\sqrt{\lambda t}$, i.e. it grows in time. The CL is the point at the velocity profile, where the flow velocity is constant and equal to the phase velocity of the wave. In this point, there is the break at the velocity profile [the scale of the break (δ) is finite but small in comparison with the diffusion length according to (13)]. This profile can be realized only if the CL is moving towards the incident wave (see Fig. 4). The similar deformation of the stratified shear flow due to nonlinear interaction with internal waves was obtained by Fritts[9] in numerical experiments. Some similar deformation of the mean flow can be seen on the photos of the laboratory experiments,[19,20] but these effects were not discussed in those papers. So for the constant jump of wave momentum flux we answered the questions formulated above, how the broken velocity profile can be realized and what the values of vorticity at the both sides of the CL are.

Now we return to the problem of interaction of the lee waves radiated from the corrugated surface and the stratified shear flow, formulated above. Suppose that the corrugated surface is placed to the undisturbed region of the stationary stratified shear flow, where the velocity profile is linear (Fig. 18). Is the jump of the wave momentum flux (and vorticity) constant? In general the wave flow interaction is the following. The wave, radiating by the surface propagates towards the DBL; partially reflects, partially transmits. Then the transmitted wave interacts with the CL: partially reflects, ab-

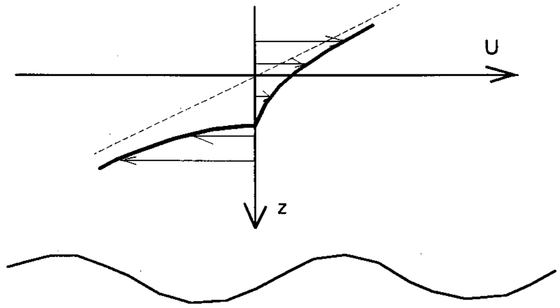

FIG. 18. The mean velocity profile forming in the flow over the corrugated surface in the presence of CL for the lee waves radiated from the surface.

sorbs and transmits. Then the transmitted wave interacts with the DBL behind the CL. The velocity profile in the DBL depends on time, because the DBL is spreading and moving. So the reflection coefficient depends on time as well. It means that the wave momentum flux, which is equal to the sum of fluxes in the incident and reflected waves, depends on time. In this case the simple self-similar velocity profile can not be obtained, but it can be found approximately for a rather wide range of parameters.

First, if the Richardson number is not very close to 1/4, then the transmitted wave behind the CL is very small. So the wave momentum flux T_- in the transmitted wave can be neglected and the jump of the wave momentum flux is approximately equal to T_+, and $\Xi = T_+/\lambda$. Second, since the time-dependence of the wave momentum flux arises due to reflection of the wave from the time-dependent velocity profile, one can expect that the time-dependent component of the wave momentum flux is small, when the reflection coefficient is small. The numerical estimations show, that when $Ri>1$ and $\lambda>0.2$, the reflection coefficient $\mathscr{R}<0.1$. In this case the solution to the equation can be sought as a series in $\mathscr{R}$. The paradoxical fact is obtained. The deformations of initially constant stratified shear flow over a corrugated surface of constant shape are time-dependent. If the reflection coefficient is not small, all above values are time-dependent as well, but analytical investigation is impossible.

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